Project
ELLEPTICAL EQUATIONS

Elliptical Equations Solutions

In the theory of partial differential equations, elliptic operators are differential operators that generalize the Laplace operator. They are defined by condition that coefficients of the highest-order derivatives be positive, which implies the key property that the principal symbol is invertible and have no real characteristic directions.

CEP 452

Computational Aspects in Water Resources

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CEP 452
ELLEPTICAL DIFFERENTIAL EQUATIONS

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PROBLEM DESCRIPTION FOR UNCONFINED AQUIFER

[*The following problem illustrates the use of the explicit and implicit methods finite difference method for evaluation of elliptical differential equations using Gauss-Seidel Method. The first problem is of an unconfined Aquifer while the second is for Confined Aquifer in Steady state.*]

A porous stratum of length 1.3 Km with permeability equal to 1cm/ sec and porosity η=0.4 with the sloping impermeable bottom is laterally confined between two water masses. The initial water level is 35m. A ditch is to be excavated at a distance of 600 m from the left reservoir and a discharge of 500 m^3/day/m is pumped from the ditch, with the simultaneous from the level of the left reservoir from 35 m to 25 m. Plot the variation of water level in the porous medium up to establishment of the steady state.

THEORY AND NUMERICAL MODELLING FOR UNCONFINED ACQUIFERS

[An unconfined aquifer is one that is open to receive water from the surface, and whose water table surface is free to fluctuate up and down, depending on the recharge/discharge rate. There are no overlying "confining beds" of low permeability to physically isolate the groundwater system.]

Used in hydrogeology, the groundwater flow equation is the mathematical relationship which is used to describe the flow of groundwater through an aquifer. The transient flow of groundwater is described by a form of the diffusion equation, similar to that used in heat transfer to describe the flow of heat in a solid (heat conduction). The steady-state flow of groundwater is described by a form of the Laplace equation, which is a form of potential flow and has analogs in numerous fields.

Mass can be represented as density times volume, and under most conditions, water can be considered incompressible (density does not depend on pressure). Using Taylor series to represent the in and out flux terms across the boundaries of the control volume, and using the divergence theorem to turn the flux across the boundary into a flux over the entire volume, the final form of the groundwater flow equation (in differential form) is:

$$S_s \frac{\partial h}{\partial t} = -\nabla \cdot q - G.$$

This is known in other fields as the diffusion equation or heat equation, it is a parabolic partial differential equation (PDE). This mathematical statement indicates that the change in hydraulic headwith time (left hand side) equals the negative divergence of the flux (q) and the source terms (G). This equation has both head and flux as unknowns, but Darcy's law relates flux to hydraulic heads, so substituting it in for the flux (q) leads to

$$S_s \frac{\partial h}{\partial t} = -\nabla \cdot (-K\nabla h) - G.$$

Especially when using rectangular grid finite-difference models (e.g. MODFLOW, made by the USGS), we deal with Cartesian coordinates. In these coordinates the general Laplacian operator becomes (for three-dimensional flow) specifically

$$\frac{\partial h}{\partial t} = \alpha \left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right] - G.$$

MODFLOW code discretizes and simulates an orthogonal 3-D form of the governing groundwater flow equation. However, it has an option to run in a "quasi-3D" mode if the user wishes to do so; in this case the model deals with the vertically averaged T and S, rather than k and Ss. In the quasi-3D mode, flow is calculated between 2D horizontal layers using the concept of leakage.

Program Developed in FORTRAN

> *[Fortran (previously FORTRAN, derived from Formula Translating System) is a general-purpose, imperative programming language that is especially suited to numeric computation and scientific computing.]*

```
implicit none

!! Declaring Variables of the program

real::t,u,k,deltat
integer::i,j,m,o
real, dimension(1000) :: Hinitial, Hfinal
real::deltax,l,dia,n,Dis,s,tot

!! Assigning Values to the variables and the Boundary Conditions

deltax=50
n=0.4
dia=20
k=864
t=0.0
Hinitial=35.0
Hfinal=35.0

!! Assigning the given data to variables according to the question

deltat=(2*n*deltax*deltax)/(2*k*dia)
Dis=-500/deltax
l = (k*deltat*dia)/(2*n*deltax*deltax)
deltat=2*deltat
m=(1300/deltax)+1
o=(600/deltax)+1

!! Calculating the head at every distance = 100m in Unconfined acquifer

100 t=t+deltat
Hinitial=Hfinal
DO j = 2,m-1
```

```
            Hfinal(j) = (l*(Hinitial(j+1)+Hinitial(j-1)-2*Hinitial(j))) + Hinitial(j)
            IF(j == o)then
            Hfinal(j)=Hfinal(j)+(Dis*deltat)/(n*deltax)
            END IF
END DO

DO j = 1, m
            s=Hinitial(j)-Hfinal(j)
            IF(s.gt.0.0) then
                        goto 100
            END IF
END DO

!! Printing Results in the File

OPEN(9,file = 'result.txt')
tot=0.0
DO j=1,m
            write(9,*)Hfinal(j)
            write(9,*)tot
            tot=tot+deltax
END DO

!! Ending the Program

STOP
END
```

Results

[The results were obtained for different time and then plotted in Microsoft Excel]

Distance(m)	Head(m)	Distance(m)	Head(m)	Distance(m)	Head(m)
0	25.10267	**450**	18.603307	**900**	25.10267
50	24.9026896	**500**	17.4034246	**950**	27.5024348
100	24.602719	**550**	16.5035128	**1000**	29.4022486
150	24.0027778	**600**	15.5036108	**1050**	31.2020722
200	23.5028268	**650**	16.603503	**1100**	32.5019448
250	22.7029052	**700**	17.9033756	**1150**	33.7018272
300	21.9029836	**750**	19.4032286	**1200**	34.3017684
350	20.9030816	**800**	21.3030424	**1250**	34.9017096
400	19.603209	**850**	23.2028562	**1300**	35.99

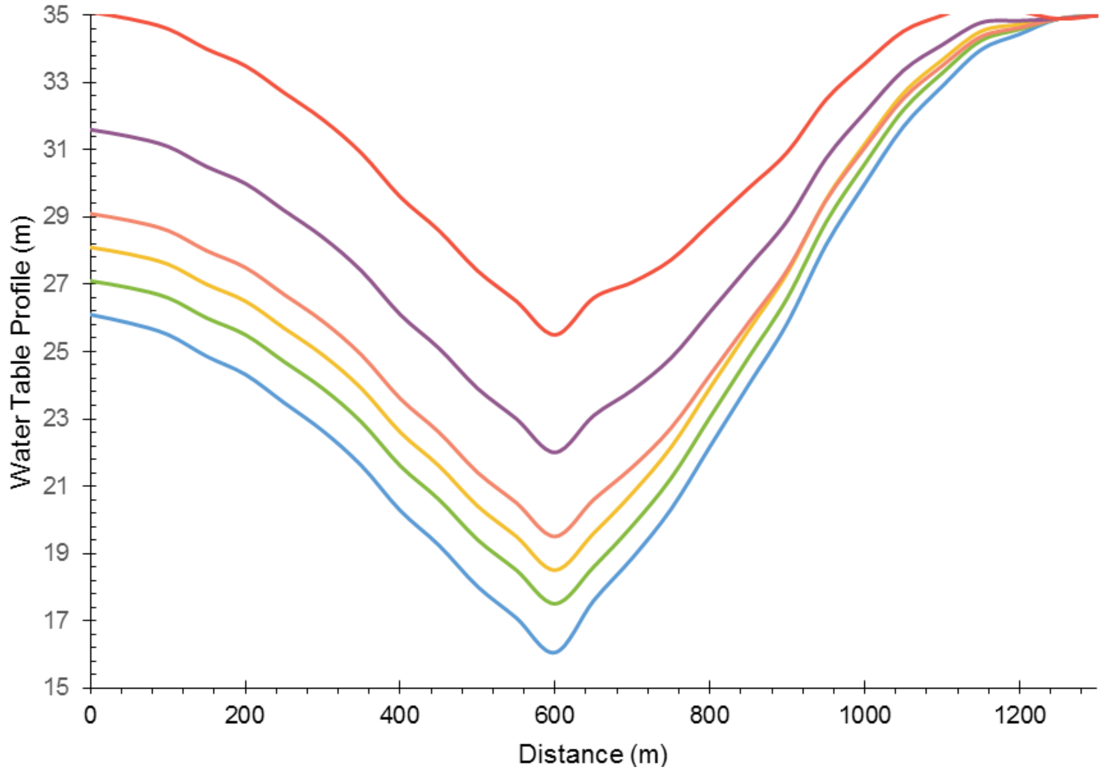


Figure 1 Time Variation of Ground Water flow head in the unconfined Aquifer

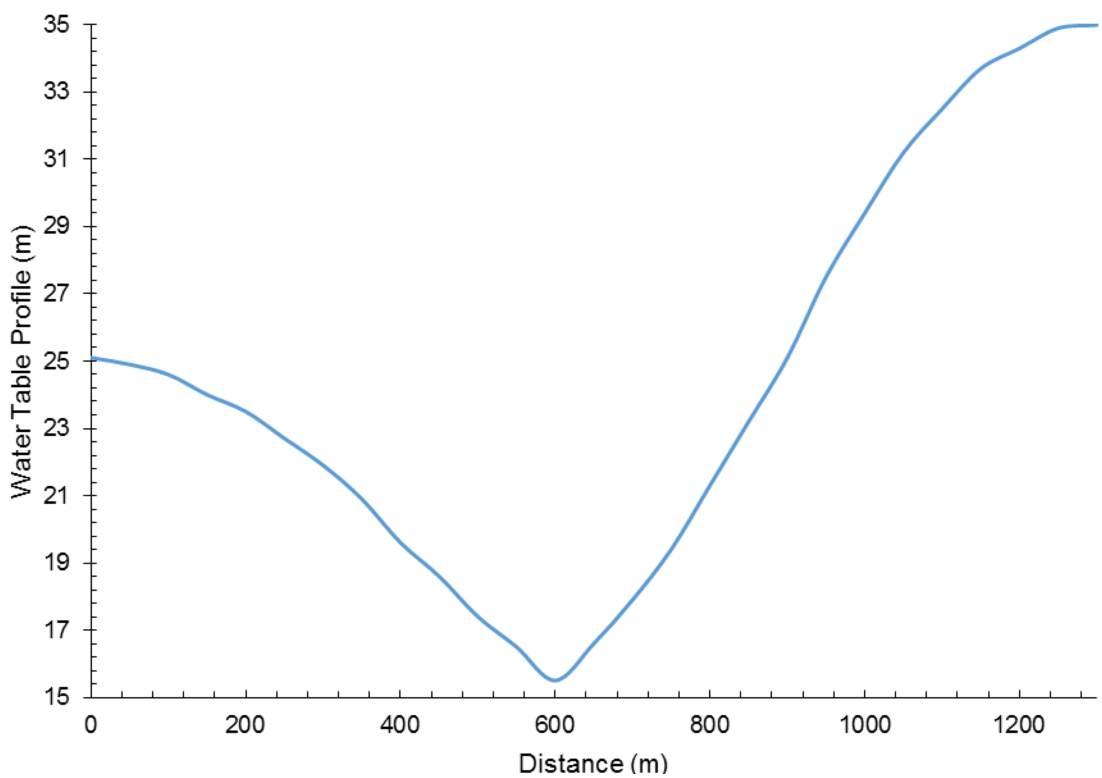


Figure 2 Steady State of Ground Water flow head in the unconfined Aquifer

PROBLEM DESCRIPTION FOR CONFINED AQUIFER

[*The following problem illustrates the use of the explicit and implicit methods finite difference method for evaluation of elliptical differential equations using Gauss-Seidel Method. The first problem is of an unconfined Aquifer while the second is for Confined Aquifer in Steady state.*]

If a building is to be founded on an island with the permeable soil. The island can be assumed to be square of side 260 m with a horizontal base of impermeable soil. The surrounding water depth is 30 m with permeability coefficient of 0.033 m/day. For the foundation of the building the water table has to be lowered from 30m to 15 m, within the base of the building i.e. a square of side 60 m located centrally on the island. This is achieved by the means of 4 wall points located at the corners of the excavation site. What is the discharge to be pumped out from each well?

THEORY AND NUMERICAL MODELLING FOR CONFINED ACQUIFERS

[A confined aquifer is a water-bearing stratum that is confined or overlain by a rock layer that does not transmit water in any appreciable amount or that is impermeable. There probably are few truly confined aquifers, because tests have shown that the confining strata, or layers, although they do not readily transmit water, over a period of time contribute large quantities of water by slow leakage to supplement production from the principal aquifer]

If the aquifer has recharging boundary conditions a steady-state may be reached (or it may be used as an approximation in many cases), and the diffusion equation (above) simplifies to the Laplace equation.

$$0 = \alpha \nabla^2 h$$

In a confined aquifer, the saturated thickness is determined by the height of the aquifer, H, and the pressure head is non-zero everywhere. In an unconfined aquifer, the saturated thickness is defined as the vertical distance between the water table surface and the aquifer base. If \partial h/\partial z=0, and the aquifer base is at the zero datum, then the unconfined saturated thickness is equal to the head, i.e., b=h.

By incorporating the correct definitions for saturated thickness, specific storage, and specific yield, we can transform this into two unique governing equations for confined and unconfined conditions:

$$S\frac{\partial h}{\partial t} = \nabla \cdot (KH\nabla h) + N.$$

(Confined), where $S=S_{sb}$ is the aquifer storativity and

$$S_y\frac{\partial h}{\partial t} = \nabla \cdot (Kh\nabla h) + N.$$

(Unconfined), where Sy is the specific yield of the aquifer.

Note that the partial differential equation in the unconfined case is non-linear, whereas it is linear in the confined case. For unconfined steady-state flow, this non-linearity may be removed by expressing the PDE in terms of the head squared:

$$\nabla \cdot (K\nabla h^2) = -2N.$$

Or, for homogeneous aquifers,

$$\nabla^2 h^2 = -\frac{2N}{K}.$$

This formulation allows us to apply standard methods for solving linear PDEs in the case of unconfined flow. For heterogeneous aquifers with no recharge, Potential flow methods may be applied for mixed confined/unconfined cases.

For solving this equation one best method is the Gauss–Seidel method, also known as the Liebmann method or the method of successive displacement, is an iterative method used to solve a linear system of equations.

Now, for each interior point (xi, yj) where i = 1, ..., n − 1 and j = 1, ..., m − 1, we can write down the finite difference equation corresponding to those points. For example, given i = 1 and j = 1, we have:

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4\,u_{1,1} = gh^2$$

In this case, two of the points are on the border (u0, 1 and u1, 0). From our boundary condition, we can find the value at these point:

$$u_{2,1} + u_{1,2} - 4\,u_{1,1} = gh^2 - \partial\mathrm{u}(x_0, y_1) - \partial\mathrm{u}(x_1, x_0)$$

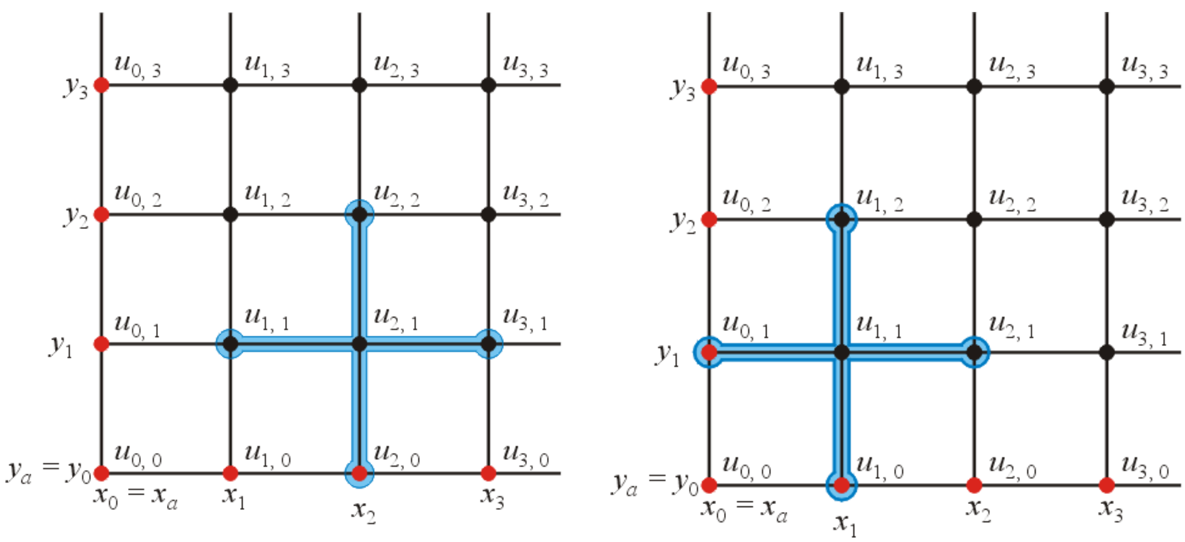
This is shown graphically in the Figure , the two border points (in red) were moved to the right-hand side of the equation.

If we repeat this process at i = 2 and j = 1, we have:

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4\,u_{2,1} = gh^2$$

In this case, only one of the points is on the border (u2, 0). From our boundary condition, we can find the value at this point:

$$u_{3,1} + u_{1,1} + u_{2,2} - 4\,u_{2,1} = gh^2 - \partial\mathrm{u}(x_2, y_0)$$



Repeating this process at each interior point, we come up with a system of (n − 1)(m − 1) linear equations in (n − 1)(m − 1) unknowns. Having such a system, we can now solve for the interior values.

Program Developed in FORTRAN

> *[Fortran (previously FORTRAN, derived from Formula Translating System) is a general-purpose, imperative programming language that is especially suited to numeric computation and scientific computing.]*

```
implicit none

!! Declaring Variables for the Program

integer :: i,j,imax,jmax,n
real :: eps, epsmax, Q,htemp,hmax
real,dimension(100,100)::h_new,h_old
OPEN(1,FILE='result.txt')

!! Assigning Values to the Variables

imax = 14
jmax = 14
n=0
Q=13.5                    !! Initial Estimate of Q
epsmax=9
h_old=0.0
DO j=1, jmax, 1
        h_old(imax,j)=900.0
END DO
DO i=1, imax, 1
        h_old(i,jmax)=900.0
END DO

!! Looping till the residue is less than 1 percent

500 DO while (epsmax > 1)
  n=n+1
        epsmax=0
        DO i=2, imax-1, 1
                DO j=2, jmax-1, 1
                        If (i==4 .AND. j==4) then
                                h_new(i,j)=(h_old(i-1,j)+h_old(i,j+1)+h_old(i+1,j)+h_old(i,j-1))/4
                                h_new=h_new + (6060.6060*Q)/          !! 2*Δx2/k=6060.6060
                        Else
                                h_new(i,j)=(h_old(i-1,j)+h_old(i,j+1)+h_old(i+1,j)+h_old(i,j-1))/4
                        END IF
                END DO
        END DO
        DO i=2, imax-1, 1
                h_new(i,1)=h_new(i,2)
        END DO
        DO j=2, jmax-1, 1
                h_new(1,j)=h_new(2,j)
        END DO
        DO i=2, imax-1, 1
                DO j=2, jmax-1, 1
```

```
                        eps=(abs(h_new(i,j)-h_old(i,j))/h_old(i,j))*100
                        IF (eps >= epsmax) then
                                epsmax=eps
                        END IF
                        h_old(i,j)=h_new(i,j)
                END DO
        END DO
END DO

!! Checking whether the Hmax in the site area is whether greater than 15m i.e 225m for H^2

DO i=1, 4, 1
        DO j=1, 4, 1
                htemp=h_new(i,j)
                IF ( htemp>225.0 ) then
                        Q= Q/10
                        Go to 500
                END IF
        END DO
END DO

Write (*,*) Q                   !! Writing the final Q result

DO i=2,imax-1,1
        h_old(i,1)=h_old(i,2)
END DO
DO j=2, jmax-1, 1
        h_old(1,j)=h_old(2,j)
END DO

!! Printing the result in the file

DO i=1, imax,1
        DO j=1, jmax,1
                WRITE (1,*) h_old(i,j)
        END DO
        Write (*,*)
END DO
Write (*,*) n                   !! Displays the no. of iterations done
END
```

Results

[The results were obtained for the steady state condition and were plotted in Microsoft Excel]

With Doing Iterative calculations for Q by increasing a little in each iteration, the value of required discharge for each of the wells at the corner of the plot was found to be about in between 14-15 m^3/day. Taking the higher and more conservative value the Q s taken as **15 m^3/day.**

In the next section I have plotted the distribution of Groundwater head with respect to distance for one symmetric section as we have evaluated in the code for. The rest plot is just symmetric to this and can be verified and evaluated similarly.

X	0	10	20	30	40	50	60	70	80	90	100	110	120	130
0	30	30	30	30	30	30	30	30	30	30	30	30	30	30
10	29.06	29.06	29.08	29.12	29.17	29.24	29.32	29.41	29.51	29.61	29.7	29.8	29.9	30
20	28.07	28.07	28.11	28.19	28.31	28.45	28.62	28.81	29	29.2	29.4	29.6	29.8	30
30	27	27	27.07	27.2	27.38	27.62	27.89	28.18	28.49	28.79	29.1	29.4	29.7	30
40	25.82	25.82	25.91	26.1	26.37	26.72	27.11	27.52	27.95	28.37	28.79	29.2	29.61	30
==50	24.48	24.48	24.6	24.86	25.24	25.73	26.27	26.83	27.4	27.95	28.49	29	29.51	30
60	22.93	22.93	23.08	23.41	23.95	24.63	25.36	26.11	26.83	27.52	28.18	28.81	29.41	30
70	21.1	21.1	21.24	21.63	22.41	23.39	24.4	25.36	26.27	27.11	27.89	28.62	29.32	30
80	18.93	18.93	18.93	19.23	20.51	22	23.39	24.63	25.73	26.72	27.62	28.45	29.24	30
90	16.49	16.49	15.93	15.25	18.07	20.51	22.41	23.95	25.24	26.37	27.38	28.31	29.17	30
100	14.26	14.26	12.31	14.14	15.25	19.23	21.63	23.41	24.86	26.1	27.2	28.19	29.12	30
110	13.65	13.65	13	12.31	15.93	18.93	21.24	23.08	24.6	25.91	27.07	28.11	29.08	30
120	13.66	13.66	13.65	14.26	16.49	18.93	21.1	22.93	24.48	25.82	27	28.07	29.06	30
130	13.67	13.66	13.65	14.26	16.49	18.93	21.1	22.93	24.48	25.82	27	28.07	29.06	30

The table below shows the Water table Head for Top-Right Quarter section of the Square Plot in the steady state. Similarly Other Too can be plotted.

In the next Section a 3-D Surface Chart has been plotted to pictorially show the Water table Depth Variation due to the Pumping Effect in the Plot.

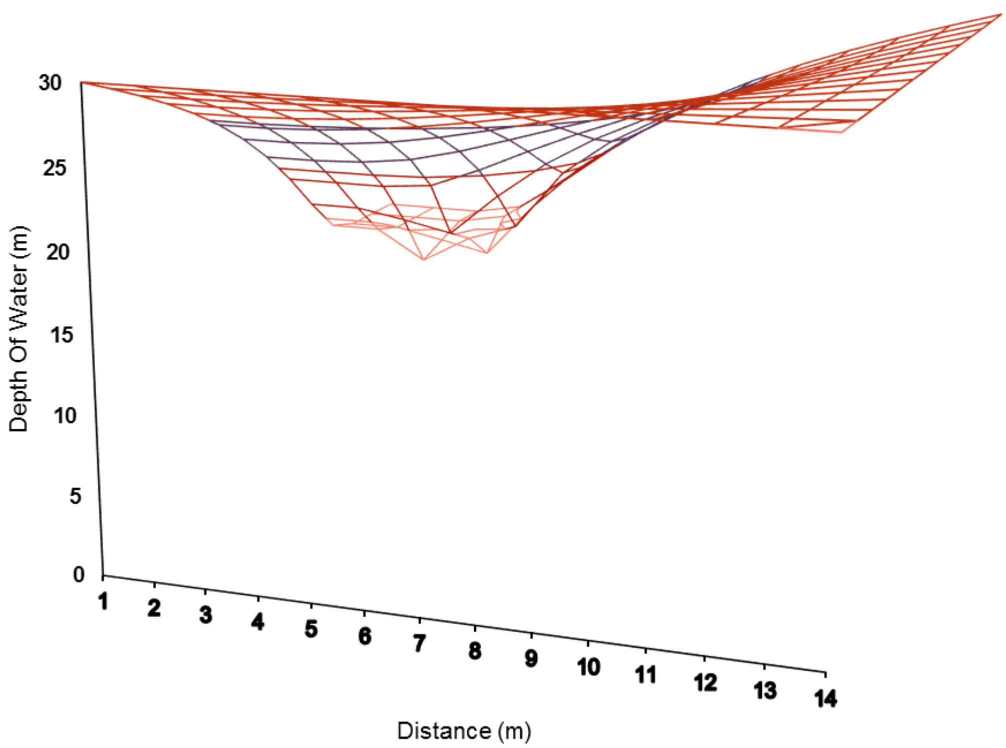


Figure 1 Front/Side View of the Top-Right Plot

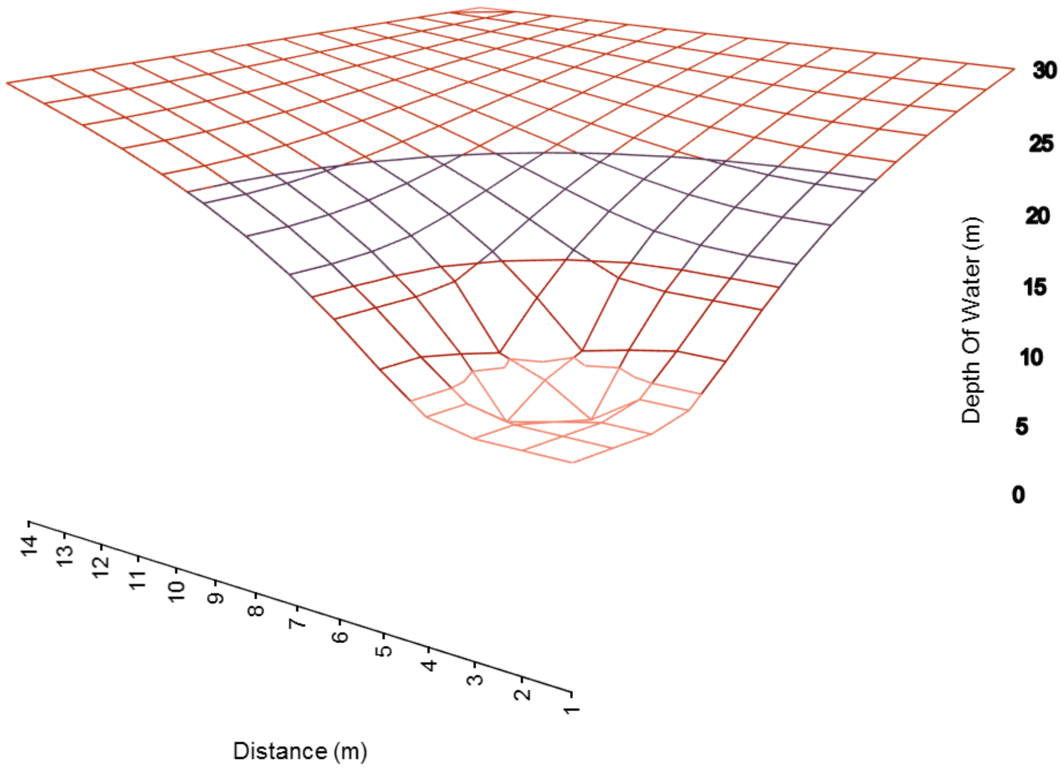


Figure 2 Top View of the Top-Right Plot